

# A continuity method to construct canonical metrics

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## 1 Introduction

The Kähler-Ricci flow has played a fundamental role in the Analytic Minimal Model Program. There has been quite a bit of progresses and many very important results have been proven (e.g., [TZ06], [ST07], [ST08] et al). In this short paper, we introduce a new continuity method which provides an alternative way of carrying out the Analytic Minimal Model Program. This method may not be as natural as the Kähler-Ricci flow, but it has the advantage of having Ricci curvature bounded from below along the deformation, so many existing analytic tools, such as the compactness theory of Cheeger-Colding-Tian for Kähler manifolds and the partial  $C^0$ -estimate, can be applied.

Assume that  $M$  is a compact Kähler manifold with a Kähler metric  $\omega_0$ . We consider the 1-parameter family of equations:

$$\omega = \omega_0 - t \operatorname{Ric}(\omega). \quad (1.1)$$

Clearly, the Kähler classes vary according to the linear relation:  $[\omega] = [\omega_0] - t c_1(M)$ , where  $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M)$  denotes the Kähler class of  $\omega$ .

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Our first theorem is:

**Theorem 1.1.** *For any initial Kähler metric  $\omega_0$ , there is a smooth family of solutions  $\omega_t$  for (1.1) on  $M \times [0, T)$ , where*

$$T = \sup\{t \mid [\omega_0] - t c_1(M) > 0\}. \quad (1.2)$$

This is an analogue of the sharp local existence theorem for the Kähler-Ricci flow due to Z. Zhang and the second author [TZ06] and it is not hard to prove.

If  $T < \infty$ , we need to examine the limit of  $\omega_t$  as  $t$  tends to  $T$ . In general, this is highly non-trivial. However, we can still prove the following:

**Theorem 1.2.** *Assume that  $T < \infty$  and  $([\omega_0] - T c_1(M))^n > 0$ , where  $n = \dim_{\mathbb{C}} M$ , then  $\omega_t$  converge to a unique weakly Kähler metric  $\omega_T$  such that  $\omega_T$  is smooth on  $M \setminus \mathcal{S}$ , where  $\mathcal{S}$  is a subvariety, and satisfies:*

$$\omega_T = \omega_0 - T \text{Ric}(\omega_T) \quad \text{on } M \setminus \mathcal{S}. \quad (1.3)$$

Furthermore,  $\mathcal{S}$  is the base locus of  $[\omega_0] - T c_1(M)$ , i.e., the set of points where  $[\omega_0] - T c_1(M)$  fails to be positive.

For any  $\omega_t$  above ( $t < T$ ), we have

$$\text{Ric}(\omega_t) = t^{-1}(\omega_0 - \omega_t) \geq t^{-1} \omega_t.$$

In particular, the Ricci curvature of  $\omega_t$  is bounded from below near  $T$ . We do expect a uniform bound on the diameter of  $\omega_t$  for any  $t \in (0, T)$  even if  $T = \infty$ . If so, by taking subsequences if necessary, we may assume that  $(M, \omega_t)$  converge to a length space  $(M_T, d_T)$  in the Gromov-Hausdorff topology. Then one should be able to further prove that  $(M_T, d_T)$  is the metric completion of  $(M \setminus \mathcal{S}, \omega_T)$ .

If  $T = \infty$ , we expect that  $\omega_t$ , after appropriate scaling, converge to a weakly Kähler-Einstein metric or a generalized Kähler-Einstein metric – introduced by Song and the second author in [ST08] – on the canonical model of  $M$ . In fact, if  $(-c_1(M))^n > 0$ , we can verify this by using the same arguments in the proof of Theorem 1.2 (see Section 3).

The organization of this paper is as follows: In the next section, we prove Theorem 1.1 by using standard arguments for complex Monge-Ampère equations. In Section 3, we prove Theorem 1.2. In Section 4, we describe the Analytic Minimal Model Program by using this new continuity method. This is parallel to what has been done for the Kähler-Ricci flow (see [Ti02], [Ti07], [ST07] and [ST08]). We will also propose a number of problems for carrying out the program.

## 2 Maximal solution time

In this section, we prove Theorem 1.1. First we reduce (1.1) to a scalar equation. Choose a real closed  $(1, 1)$  form  $\psi$  representing  $c_1(X)$  and a smooth volume form

$\Omega$  such that  $\text{Ric}(\Omega) = \psi$ . This  $\Omega$  is unique up to multiplication by a positive constant.

Set  $\tilde{\omega}_t = \omega_0 - t\psi$  for  $t \in [0, T)$ . One can easily show that  $\omega = \tilde{\omega}_t + t\sqrt{-1}\partial\bar{\partial}u$  satisfies (1.1) if  $u$  satisfies

$$(\tilde{\omega}_t + t\sqrt{-1}\partial\bar{\partial}u)^n = e^u \Omega, \quad (2.1)$$

where  $\tilde{\omega}_t + t\partial\bar{\partial}u > 0$ . This equation depends on the choice of  $\psi$ , but if we choose a different representative  $\hat{\psi}$  of  $c_1(M)$ , (2.1) changes in a simple way: Write  $\hat{\psi} = \psi + \sqrt{-1}\partial\bar{\partial}v$  and  $\hat{\omega}_t = \omega_0 - t\hat{\psi}$ , then for any solution  $u$  of (2.1),  $\hat{u} = u - v$  solves

$$(\hat{\omega}_t + t\sqrt{-1}\partial\bar{\partial}\hat{u})^n = e^{\hat{u}} \hat{\Omega}, \quad (2.2)$$

where  $\hat{\Omega}$  is a volume form satisfying:

$$\text{Ric}(\hat{\Omega}) = \hat{\psi}, \quad \int_M e^u \Omega = \int_M e^{\hat{u}} \hat{\Omega}.$$

This shows that the solvability of (2.1) is independent of the choice of  $\psi$ .

On the other hand, it follows from the definition of  $T$  that for any  $\bar{t} < T$ , there is a  $\psi$  such that  $\omega_0 - \bar{t}\psi > 0$ . Thus, in order to prove Theorem 1.1, we only need to prove that (2.1) is solvable for such a  $\psi$  and any  $t \in [0, \bar{t}]$ . Put

$$E = \{t \in [0, \bar{t}] \mid (2.1) \text{ has a solution} \}.$$

Clearly,  $0 \in E$  since  $u = \log\left(\frac{\omega_0^n}{\Omega}\right)$  is an obvious solution. So  $E$  is non-empty.

**Lemma 2.1.** *The set  $E$  is open.*

*Proof.* Assume that  $t_1 \in E$  and  $u_1$  be a solution of (2.1) with  $t = t_1$ . We want to solve (2.1) for  $t$  close to  $t_1$ . If  $t_1 > 0$ , that is readily done. Write  $tu = t_1 u_1 + w$  for some small  $w$ . Then (2.1) becomes

$$(\omega_1 - (t - t_1)\psi + \sqrt{-1}\partial\bar{\partial}w)^n = e^{\frac{w}{t} - \frac{u_1(t-t_1)}{t}} \omega_1^n,$$

where  $\omega_1 = \omega_0 - t_1\psi + t_1\sqrt{-1}\partial\bar{\partial}u_1$ . In fact,  $\Omega = e^{-u_1}\omega_1^n$  and  $\omega_t = \omega_0 - t_1\psi - (t - t_1)\psi + \sqrt{-1}\partial\bar{\partial}(t_1 u_1 + w) = \omega_1 - (t - t_1)\psi + \sqrt{-1}\partial\bar{\partial}w$  and finally we have used that  $e^u \Omega = e^{\frac{t_1 u_1 + w}{t} - u_1} \omega_1^n$ . Naturally, one still has that  $\omega_1 - (t - t_1)\psi > 0$  for  $t - t_1$  sufficiently small and therefore, setting  $\hat{\omega}_1 := \omega_1 - (t - t_1)\psi > 0$  and choosing  $F$  such that  $\hat{\omega}_1^n = e^F \omega_1^n$ , one can write the equation above as:

$$(\hat{\omega}_1 + \sqrt{-1}\partial\bar{\partial}w)^n = e^{\frac{w}{t} - \frac{u_1(t-t_1)}{t} - F} \hat{\omega}_1^n,$$

From now on, for the sake of notation we shall write  $\omega_1$  instead of  $\hat{\omega}_1$ .

Expanding in  $w$ , we get

$$\Delta_1 w - t^{-1}w = \frac{u_1(t_1 - t)}{t} + Q(\nabla^2 w),$$

where  $\Delta_1$  denotes the Laplacian of  $\omega_1$  and  $Q(a)$  denotes a polynomial in  $a$  starting with a quadratic term. Applying the Implicit Function Theorem, one can easily solve the above equation for  $w$ , consequently  $u$ , for  $t - t_1$  sufficiently small.

If  $t_1 = 0$ , we need to work a bit more carefully since the left side of (2.1) is degenerate. Write

$$u = \log\left(\frac{\omega_0^n}{\Omega}\right) + t^{-1}w.$$

Then (2.1) becomes

$$\Delta_0 w - t^{-1}w = -t \Delta_0 \log\left(\frac{\omega_0^n}{\Omega}\right) + Q\left(t \nabla \log\left(\frac{\omega_0^n}{\Omega}\right) + \nabla^2 w\right), \quad (2.3)$$

where  $\Delta_0$  is the Laplacian of  $\omega_0$ . Put

$$A = \left\| \Delta_0 \log\left(\frac{\omega_0^n}{\Omega}\right) \right\|_{C^{\frac{1}{3}}}.$$

**Claim:** There is a uniform constant  $C$  such that for any  $f \in C^{\frac{1}{3}}$ , there is a solution  $v$  satisfying:

$$\Delta_0 v - t^{-1}v = tf \quad \text{and} \quad t^{-1}\|v\|_{C^0} + \|v\|_{C^{2,\frac{1}{3}}} \leq Ct^{\frac{2}{3}}\|f\|_{C^{\frac{1}{3}}}.$$

*Proof.* Let us prove this claim. First by the Maximum Principle,  $|v| \leq At^2$ , with  $A := \|f\|_{\infty}$ . Then by standard elliptic theory,  $\|v\|_{C^1} \leq C't$ . We can deduce from these:  $t^{-1}\|v\|_{C^{\frac{1}{3}}} \leq C''t^{\frac{2}{3}}$ <sup>1</sup>. Then by elliptic theory again, we get  $\|v\|_{C^{2,\frac{1}{3}}} \leq C'''At^{\frac{2}{3}}$ . The claim is proved.  $\square$

Now we can complete the proof of this lemma by standard iteration: Set  $w_0 = 0$  and construct  $w_i$  for  $i \geq 1$  by solving the equation:

$$\Delta_0 w_i - t^{-1}w_i = -t \Delta_0 \log\left(\frac{\omega_0^n}{\Omega}\right) + Q\left(t \nabla \log\left(\frac{\omega_0^n}{\Omega}\right) + \nabla^2 w_{i-1}\right). \quad (2.4)$$

If  $\|w_{i-1}\|_{C^{2,\frac{1}{3}}} \leq C(1+A)t^{\frac{2}{3}}$ , then for  $t$  sufficiently small, the right side of (2.4) is bounded by  $At$ , so by the above claim, we get

$$\|w_i\|_{C^{2,\frac{1}{3}}} \leq C(1+A)t^{\frac{2}{3}}.$$

Moreover, by the above claim, we have

$$t^{-\frac{2}{3}}\|w_{i+1} - w_i\|_{C^{2,\frac{1}{3}}} \leq Ct^{\frac{1}{3}}\left(t^{-\frac{2}{3}}\|w_i - w_{i-1}\|_{C^{2,\frac{1}{3}}}\right).$$

It follows that for sufficiently small  $t$ ,  $w_i$  converge to a  $C^2$ -function  $w$  which solves (2.3), so we get a solution for (2.1) for  $t$  sufficiently small, i.e.,  $E$  is open.  $\square$

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<sup>1</sup>Both  $C'$  and  $C''$  are uniform constants.

It remains to prove that  $E$  is closed. Assume that  $\{t_i\} \subset E$  is a sequence with  $\lim t_i = \bar{t} > 0$  and  $u_i$  is the solution of (2.1) with  $t = t_i$ . We want to prove  $\bar{t} \in E$ . It amounts to getting an a priori  $C^{2,1/2}$ -estimate for  $u_i$ . By applying the Maximum Principle to (2.1), we have

$$\sup_M |u_i| \leq \sup_M \left| \log \left( \frac{\omega_0^n}{\Omega} \right) \right|.$$

So we have the  $C^0$ -estimate. For the  $C^2$ -estimate, it is easier to use the Schwartz-type estimate.

Using (2.1) or equivalently, (1.1), we have

$$\text{Ric}(\omega_i) = \frac{1}{t_i}(\omega_0 - \omega_i) \geq -\frac{1}{t_i}\omega_i,$$

where  $\omega_i = \tilde{\omega}_{t_i} + t_i \sqrt{-1} \partial \bar{\partial} u_i$ . then by standard computations, we have

$$\Delta_i \log \text{tr}_{\omega_i}(\tilde{\omega}_{t_i}) \geq -a \text{tr}_{\omega_i}(\tilde{\omega}_{t_i}) - \frac{1}{t_i},$$

where  $\Delta_i$  is the Laplacian of  $\omega_i$  and  $a_i$  is a positive upper bound of the bisectional curvature of  $\tilde{\omega}_{t_i}$ . Since  $\lim t_i = \bar{t}$  and  $\{\tilde{\omega}_t\}$  is a smooth family of Kähler metrics for  $t$  near  $\bar{t}$ , we have  $a = \sup_i a_i < \infty$ .

If we put

$$v = \log \text{tr}_{\omega_i}(\tilde{\omega}_{t_i}) - (a+1)t_i u_i,$$

then it follows

$$\Delta_i v \geq e^{v-(a+1)t_i c} - n(a+1) - \frac{1}{t_i},$$

where  $c = \sup_i (-\inf_M u_i)$ . Hence, by using the Maximum Principle, we can bound  $v$  from above, so there is a uniform constant  $C$  such that

$$C^{-1} \omega_0 \leq \omega_i.$$

Using (2.1), we derive

$$C^{-1} \tilde{\omega}_{t_i} \leq \omega_i \leq C \tilde{\omega}_{t_i}. \quad (2.5)$$

Next, by applying Calabi's 3rd derivative estimate, (cf. [Ya78])<sup>2</sup> for a uniform constant  $C'$ , we have

$$\|u_i\|_{C^3} \leq C'.$$

Thus, by taking a subsequence if necessary,  $u_i$  converge to a  $C^3$ -solution of (2.1) with  $t = \bar{t}$ , i.e.,  $\bar{t} \in E$  and  $E$  is closed. Theorem 1.1 is proved.

**Corollary 2.2.** *If  $K_M$  is nef, then (2.1) has a unique solution for any  $t > 0$ .*

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<sup>2</sup>One may also use the  $C^{2,\alpha}$ -estimate (cf. [Ev82], [Ti84]).

### 3 Proof of Theorem 1.2

First we observe

**Lemma 3.1.** *For any solution  $u$  of (2.1) with  $t \in (0, T)$ , we have*

$$\sup_M u \leq \sup_M \log \left( \frac{\omega_t^n}{\Omega} \right),$$

*i.e.,  $u$  is uniformly bounded from above.*

The proof of Theorem 1.2 is now quite standard. For simplicity, we assume that  $M$  is a projective manifold and  $[\omega_0] = 2\pi c_1(L)$  for some line bundle  $L$ . Under this assumption, we can use the following lemma due to Kodaira (cf. [Ka84]). The general case can be proved in an identical way if we use instead Demailly-Paun's extension [DP04] of Kodaira's lemma to Kähler manifolds.

**Lemma 3.2.** *Let  $E$  be a divisor in a projective manifold  $M$ . If  $E$  is nef and big, then there is an effective  $\mathbb{R}$  divisor  $D = \sum_i a_i D_i$ , where  $D_i$  are divisors and  $a_i$  are positive real numbers, such that  $E - [D] > 0$ , where  $[D] = \sum_i a_i [D_i]$  and each  $[D_i]$  denotes the line bundle induced by  $D_i$ .*

This follows from the openness of the big cone of  $M$  which clearly contains the positive cone and the fact that  $E$  is in the closure of the positive cone. Recall that the non-ample locus  $B_+(E)$  of  $E$  is defined to be the intersection of  $\text{Supp}(D)$ , where  $D$  ranges over all effective  $\mathbb{R}$ -divisor such that  $E - D$  is ample. The above lemma implies that  $B_+(E)$  is a subvariety whenever  $E$  is big.

Now we take  $E = L + TK_M$  which is nef and big, so by the above lemma, for any  $x \in M \setminus B_+(E)$ , there is an effective divisor  $D = \sum_i a_i D_i$  as above such that  $L + TK_M - [D]$  is ample.

For each  $i$ , let  $\sigma_i$  be the defining section of  $D_i$  and  $\|\cdot\|_i$  be a Hermitian norm on  $[D_i]$ . Then  $\sum_i a_i \log \|\sigma_i\|_i^2$  is a well-defined function outside  $\text{Supp}(D) \subset M$ . Since  $L + TK_X - [D]$  is ample, we can choose  $\|\cdot\|_i$  such that

$$\tilde{\omega}_T + \sqrt{-1} \sum_i a_i \partial \bar{\partial} \log \|\sigma_i\|_i^2 > 0.$$

For simplicity, we will write

$$\log \|\sigma\|^2 = \sum_i a_i \log \|\sigma_i\|_i^2.$$

Formally, we regard  $\sigma$  as a defining section of  $D$  and  $\|\cdot\|$  as a norm on  $[D]$ .

Set

$$\tilde{\omega}_{t,D} = \tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \log \|\sigma\|^2.$$

Then there is a  $\delta = \delta(D)$ , which may depend on  $D$ , such that  $\omega_{t,D}$  is a smooth Kähler metric for any  $t \in [T - \delta, T + \delta]$ . For any solution  $u$  of (2.1) with  $t \in [T - \delta, T]$ , we put

$$v = tu - \log \|\sigma\|^2,$$

then  $\omega_t = \tilde{\omega}_{t,D} + \sqrt{-1} \partial \bar{\partial} v$  and  $v$  satisfies the following equation:

$$(\tilde{\omega}_{t,D} + \sqrt{-1} \partial \bar{\partial} v)^n = e^{\frac{1}{t}(v + \log \|\sigma\|^2)} \Omega. \quad (3.1)$$

Applying the Maximum Principle to (3.1), we get  $v \geq -c$  for a uniform constant which may depend on  $D$ . So by changing  $c$  if necessary, we have

$$-c \leq v \leq c - \log \|\sigma\|^2.$$

In particular,  $v$  or equivalently  $u$ , is bounded outside  $D$ . Since  $\text{Ric}(\omega_t)$  is bounded below by  $-1/t$ , as in last section, we can infer

$$\Delta_t \log \text{tr}_{\omega_t}(\tilde{\omega}_{t,D}) \geq -a \text{tr}_{\omega_t}(\tilde{\omega}_{t,D}) - \frac{1}{t},$$

where  $\Delta_t$  denotes the Laplacian of  $\omega_t$  and  $a$  is a positive upper bound on the bisectional curvature of  $\tilde{\omega}_{t,D}$  for all  $t \in [T - \delta, T]$ . Put

$$w = \log \text{tr}_{\omega_t}(\tilde{\omega}_{t,D}) - (a + 1)v.$$

Then we have

$$\Delta_t w \geq e^{w - (a+1)b} - n(a+1) - \frac{1}{t},$$

where  $b = -\inf_M v$ . Hence, by using the Maximum Principle, we can bound  $w$  from above, so there is a uniform constant  $C$  such that

$$C^{-1} \|\sigma\|^{2(a+1)} \tilde{\omega}_{t,D} \leq \omega_t.$$

Using (3.1), we derive

$$C^{-1} \|\sigma\|^{2(a+1)} \tilde{\omega}_{t,D} \leq \omega_t \leq C \|\sigma\|^{-2(n-1)(a+1) + \frac{1}{t}} \tilde{\omega}_{t,D}. \quad (3.2)$$

Then by Calabi's 3rd derivative estimate, for any compact subset  $K \subset M \setminus D$ , we have  $\|u\|_{C^3(K)} \leq C_K$  for some uniform constant  $C_K$ . It follows that any sequence  $\{t_i\}$  with  $t_i \rightarrow T$  has a subsequence, still denoted by  $t_i$  for simplicity, such that  $v_{t_i}$  converge a  $C^3$ -function  $v_t$  on  $M \setminus D$  satisfying:

$$(\tilde{\omega}_{T,D} + \sqrt{-1} \partial \bar{\partial} v_T)^n = e^{\frac{1}{T}(v_T + \log \|\sigma\|^2)} \Omega. \quad (3.3)$$

A priori, this limit may not be unique. so we still need to prove that  $v_T$  is unique, i.e., independent of the sequence  $\{t_i\}$ .

**Lemma 3.3.** *Let  $\dot{v}$  be the derivative of  $v$  in the  $t$ -direction and  $T < \infty$ . Then there is a uniform constant  $C_T$ , which may depend on  $T$ , such that*

$$\int_M |\dot{v}|^2 \omega_t^n \leq C_T \quad \text{on } t \in (T - \delta, T). \quad (3.4)$$

*Proof.* Differentiating (3.1) on  $t$ , we get

$$\Delta_t \dot{v} = -\frac{1}{t^2} u + \frac{1}{t} \dot{v}.$$

Note that  $\dot{v}$  is equal to  $t\dot{u} + u$  which is smooth on  $M \times (0, T)$  since  $\log \|\sigma\|^2$  is independent of  $t$ . Since  $u$  is bounded from above, we deduce from (2.1)

$$\int_M |u|^2 \omega_t^n = \int_M u^2 e^u \Omega \leq C,$$

where  $C$  is a uniform constant. Then we have

$$\int_M \left( |\nabla \dot{v}|^2 + \frac{1}{t} |\dot{v}|^2 \right) \omega_t^n = \frac{1}{t^2} \int_M \dot{v} u \omega_t^n. \quad (3.5)$$

Then (3.4) follows easily from this and the Cauchy inequality.  $\square$

For any compact subset  $K \subset M \setminus D$ , by (3.2), we have that for some  $C_K > 0$ ,

$$C_K^{-1} \omega_0 \leq \omega_t \leq C_K \omega_0.$$

Therefore, we have for  $T - \delta < t < t' < T$ ,

$$\int_K |v(x, t) - v(x, t')|^2 \omega_0^n(x) \leq \int_t^{t'} \int_K |\dot{v}(\cdot, s)|^2 \omega_0^n ds \leq C_T (C_K)^n |t' - t|.$$

It follows that  $v(\cdot, t)$  and  $v(\cdot, t')$  converge to the same function  $v_T$  on  $M \setminus D$ .

Since  $D$  is any divisor in the definition of  $B_+(E)$ , where  $E = L + T K_M$ , we have proved Theorem 1.2 with  $\mathcal{S} = B_+(E)$ .

## 4 Analytic Minimal Model Program revisited

In this section, we follow the lines of the approach towards the Analytic Minimal Model Program through Ricci flow (cf. [Ti07], [ST07], [ST08] et al) to list some problems and speculations. Some of these problems are doable by adapting arguments from what has been done for the Kähler-Ricci flow (cf. above citations). It is possible to get even stronger results because Ricci curvature is bounded from below in this new approach through the continuity method.

Let  $u_t$  be the maximal solution of (2.1) for  $t \in (0, T)$  and write

$$\omega_t = \omega_0 - t\psi + \sqrt{-1} \partial \bar{\partial} u_t.$$

First we assume  $T < \infty$ .

**Conjecture 4.1.** *As  $t \rightarrow T$ ,  $(M, \omega_t)$  converges to a compact metric space  $(M_T, d_T)$  in the Gromov-Hausdorff topology<sup>3</sup> satisfying the following:*

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<sup>3</sup>Since  $\omega_t$  has Ricci curvature bounded from below, this is equivalent to that the diameter of  $(M, \omega_t)$  is uniformly bounded.



- (1)  $M_T$  is a Kähler variety and there is a holomorphic fibration  $\pi_T : M \mapsto M_T$ ;
- (2)  $d_T$  is a “nice” Kähler metric  $\omega_T$  on  $M_T \setminus \mathcal{S}_T$ , where  $\mathcal{S}_T$  is a subvariety of  $M_T$  containing all the singular points. If  $([\omega_0] - Tc_1(M))^n > 0$ , then it is the same as saying that  $\omega_T$  is the limit given in Theorem 1.2 and  $(M_T, d_T)$  is the metric completion of  $(M \setminus \mathcal{S}, \omega_T)$ ;
- (3)  $\omega_t$  converge to  $\omega_T$  on  $\pi^{-1}(M_T \setminus \mathcal{S}_T)$  in a much regular topology, possibly, the smooth topology.

It seems that the key for solving this conjecture is to bound the diameter of  $(M, \omega_t)$ , especially, in the non-collapsing case.

Next we want to examine how to extend our continuity method beyond  $T$ . To this end, we propose:

**Conjecture 4.2.** *The variety  $M_T$  has a partial resolution  $\pi'_T : M' \mapsto M_T$ , which may have “mild” singularity, such that the canonical sheaf  $K_{M'}$  is well-defined and  $\pi'_T(\pi_T[\omega_T]) + tK_{M'} > 0$  for  $t > 0$  small. If  $\dim M_T = \dim M$ , i.e., in the non-collapsing case,  $M'$  should be a flip of  $M$  as defined in algebraic geometry.*

This may be the most difficult part of the program as we have learned from the Analytic Minimal Model Program through Ricci flow. More precisely, by “mild” singularity, we mean the following

**Conjecture 4.3.** *Let  $\omega'_0 = \pi'^*_T(\pi_{T*}\omega_T)$ . Then we can solve (2.1) on  $M'$  for  $t \in (0, T')$  with initial metric  $\omega'_0$ , where*

$$T' = \sup\{t \mid \pi'^*_T(\pi_{T*}[\omega_T]) + tK_{M'} > 0\}.$$

By the work in [ST08], one knows that log-terminal singularities have the property in the conjecture above, for the Ricci flow. The same should be true for the new continuity method. In view of [ST08], it may be possible to solve Conjecture 4.3 by the current technology at hand.

If the above conjectures can be affirmed, for some initial Kähler metric  $\omega_0$  on  $M$ , we can construct a family of pairs  $(M_t, \omega_t)$  ( $0 \leq t < \infty$ ) together with a sequence of times  $T_0 = 0 < T_1 < T - 2 < \dots < T_k < \dots$ , satisfying:

- (1) For  $t \in [T_i, T_{i+1})$  ( $i \geq 0$ ),  $M_t = M_i$  is a fixed Kähler variety<sup>4</sup> and  $\omega_t$  is a solution of (1.1) on  $M_i$  in a suitable sense. Moreover,  $M_0 = M$ ;
- (2) For  $i \geq 1$ ,  $M_{i+1}$  is a “flip” of  $M_i$  as described in Conjecture 4.2. If we denote by  $\pi_i : M_i \mapsto M_{T_i}$  and  $\pi_{i+1} : M_{i+1} \mapsto M_{T_i}$  the natural projections in the flip process, then

$$\lim_{t \rightarrow T_i^-} \pi_i^* \omega_t = \lim_{t \rightarrow T_i^+} \pi_{i+1}^* \omega_t.$$

We also have  $\lim_{t \rightarrow 0} \omega_t = \omega_0$ ;

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<sup>4</sup> $M_i$  may be of smaller dimension or even an empty set.

(3)  $\omega_t$  are smooth on the regular part of  $M_t$  and continuous on the level of potentials along  $t$ .

Such a family corresponds to the solution of the Kähler-Ricci flow with surgery.

As in the case of Kähler-Ricci flow, we call  $T_i$  a surgery time. We expect that for each initial Kähler metric  $\omega_0$ , there are only finitely many surgery times, that is,

**Conjecture 4.4.** *There are only finitely many surgery times  $T_0 = 0 < T_1 < T_2 < \dots < T_N < \infty$  such that  $M_t = M_N$  for  $t > T_N$  is either empty or a minimal model of  $M$ . If  $M_N \neq \emptyset$ , then  $K_{M_N}$  is nef, consequently, (1.1) admits a solution  $\omega_t$  for all  $t > T_N$ . If  $M_N = \emptyset$ ,  $M$  is birational to a Fano-like manifold and the converse is also true.*

Next, assuming that  $M$  is a minimal model, smooth or with “mild” singularities, we need to analyze the asymptotic of solutions  $\omega_t$  of (1.1) as  $t$  tends to  $\infty$ .

**Conjecture 4.5.** *If the Kodaira dimension  $\kappa(M) = 0$ , then  $\omega_t$  should converge to a Calabi-Yau metric  $\omega_\infty$  on  $M \setminus D$ , where  $D$  is a subvariety, in the smooth topology. Furthermore,  $(M, \omega_t)$  should converge to a compact metric space  $(M_\infty, d_\infty)$  in the Gromov-Hausdorff topology such that  $M_\infty$  is the metric completion of  $(M \setminus D, \omega_\infty)$  and  $d_\infty|_{M \setminus D}$  is induced by  $\omega_\infty$ .*

Similarly, we expect

**Conjecture 4.6.** *If  $\kappa(M) = \dim M = n$ , then  $t^{-1}\omega_t$  should converge to a Kähler-Einstein metric  $\omega_\infty$  with scalar curvature  $-n$  on  $M \setminus D$ , where  $D$  is a subvariety, in the smooth topology. Furthermore,  $(M, t^{-1}\omega_t)$  should converge to a compact metric space  $(M_\infty, d_\infty)$  in the Gromov-Hausdorff topology such that  $M_\infty$  is the metric completion of  $(M \setminus D, \omega_\infty)$  and  $d_\infty|_{M \setminus D}$  is induced by  $\omega_\infty$ .*

In view of recent works of Jian Song [So14], we believe that the above two conjectures are solvable. The key should be a diameter estimate on  $(M, \omega_t)$ .

It remains to consider the cases:  $1 \leq \kappa(M) \leq n - 1$ . In these cases, we can not expect the existence of any Kähler-Einstein metrics (even with possibly singular along a subvariety) on  $M$  since  $K_M^n = 0$ . However, we expect

**Conjecture 4.7.** *If  $1 \leq \kappa(M) \leq n$ , then  $(M, t^{-1}\omega_t)$  should converge to a compact metric space  $(M_\infty, d_\infty)$  in the Gromov-Hausdorff topology such that  $M_\infty$  is a Kähler variety of complex dimension  $\kappa(M)$  and  $d_\infty$  is induced by a generalized Kähler-Einstein metric  $\omega_\infty$  (cf. [ST08]) on the regular part of  $M_\infty$ . Moreover,  $(M, t^{-1}\omega_t)$  should converge to  $\omega_\infty$  in a stronger topology, such as in  $C^{1,1}$ -topology on Kähler potentials, on the regular part of  $M_\infty$ .*

The existence of generalized Kähler-Einstein metrics was established in [ST08].

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